Slowly rotating compact stars

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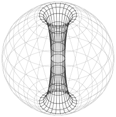


Introduction

- SLOWLY ROTATING

$$\Omega^2 \ll GM/R^3$$

for typical values $M = 1.4M_\odot,\ R = 10\ \mathrm{km}$ the relation gives

$$(f/2200\mathrm{Hz})^2 \ll 1$$

Fastest pulsar (ATNF Pulsar Catalogue) $f = 716\ \mathrm{Hz}$

- COMPACT STARS
  - Neutron stars - mixture of $n,\ p,\ e^-$ with μ (with possible hyperon $\Lambda,\ \Sigma^{\pm,0}, \Delta^{\pm,0}$, or meson $\pi,\ \kappa$ condensates)
  - Strange stars - mixture of equal number of u, d, s quarks, could be bare or with crust
  - Hybrid stars - neutron stars with core made of strange matter

Equation of state of nuclear matter

- Different equations of state of nuclear matter are represented by different $E = E(n_{\rm b}, x_{\rm p})$ relations which follows directly the theory and the method that is used
  - E... binding energy per baryon
  - $n_{\rm b}$... baryon number density
  - $x_{\rm p} = n_{\rm p}/n_{\rm b}$... proton fraction
- The relation $E = E(n_{\rm b}, x_{\rm p})$ is very well approximated by

$$E(n_{\rm b}, x_{\rm p}) = E_{\rm SNM}(n_{\rm b}) + \delta^2 S(n_{\rm b}),$$

where $\delta = 1 - 2x_{\rm p} = (n_{\rm n} - n_{\rm p})/n_{\rm b}$ is the asymmetry and the $S(n_{\rm b}) = E_{\rm PNM}(n_{\rm b}) - E_{\rm SNM}(n_{\rm b})$ is the symmetry energy.

Equation of state of nuclear matter

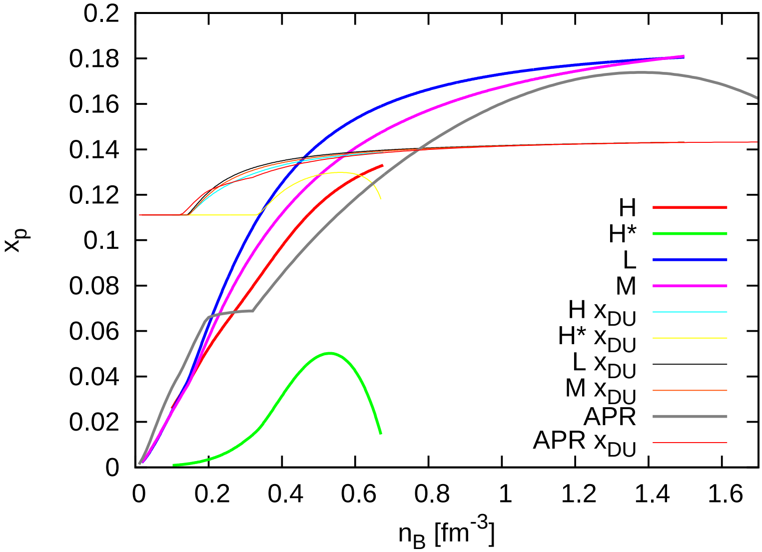
- Matter inside neutron stars is in the $\beta-$ equilibrium i.e. in equilibrium with respect to reactions $\mathrm{n} \leftrightarrow \mathrm{p} + \mathrm{e}^- \leftrightarrow \mathrm{p} + \mu$
- The condition of β-equilibrium follows the quadratic dependency of energy on asymmetry parameter and reads

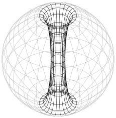
$$\mu_{\mathrm{e}} = \mu_{\mu} = \mu_{\mathrm{n}} - \mu_{\mathrm{p}} = 4\frac{S(n_{\mathrm{b}})}{n_{\mathrm{b}}}\left(1 - 2x_{\mathrm{p}}\right).$$

- β-equilibrium condition is solved together with condition of charge neutrality ($n_{\mathrm{p}} = n_{\mathrm{e}} + n_{\mu}$) to obtain the proton fraction of neutron star matter for different baryon number densities n_{b}.
- The pressure is then calculated from the relation

$$P = n_{\mathrm{b}}^2 \frac{\partial}{\partial n_{\mathrm{b}}} E(n_{\mathrm{b}}, x_{\mathrm{p}})$$

Proton fraction of nuclear matter at β-equilibrium





Equation of state of strange matter

- Strange stars consist of equal number of u, d and s quarks.
- The simplest model is the MIT Bag Model given by the Bag constant B and strong interaction coupling constant α_{C}
- We use MIT Bag Model with $B = 10^{14}\mathrm{g.cm}^{-3}$ and $\alpha_{\mathrm{C}} = 0.15$

$$P = \frac{1}{3}\left(\rho - 4B\right)$$

$$n_{\mathrm{B}} = \left[\frac{4(1 - 2\alpha_{\mathrm{c}}/\pi)^{1/3}}{9\pi^{2/3}\hbar}(\rho - B)\right]^{3/4}$$

Model of Static Non - Rotating NS

Spherical spacetime is assumed

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu_0}\mathrm{d}t^2 + \mathrm{e}^{2\lambda_0}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

where

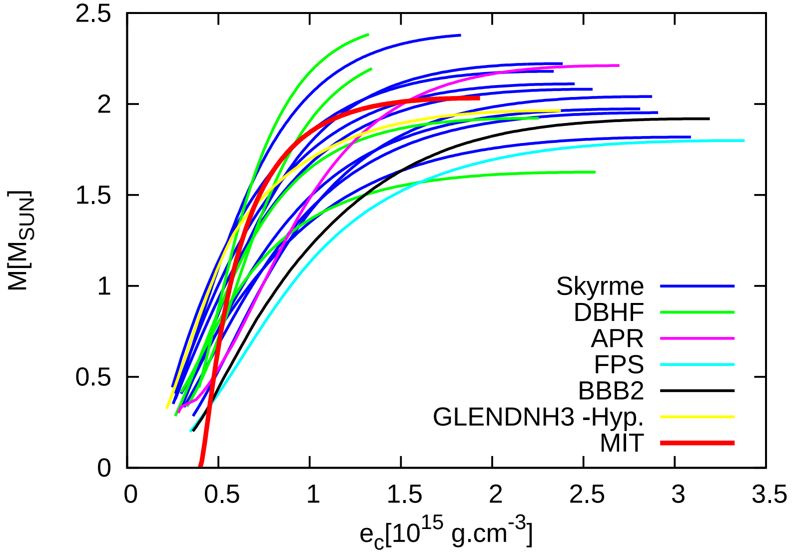
$$\mathrm{e}^{2\lambda_0} = \frac{r}{r - 2m(r)}, \quad m(r) = 4\pi\int_0^r \rho r^2 \mathrm{d}r.$$

The equation of hydrostatic equilibrium is given by TOV equation

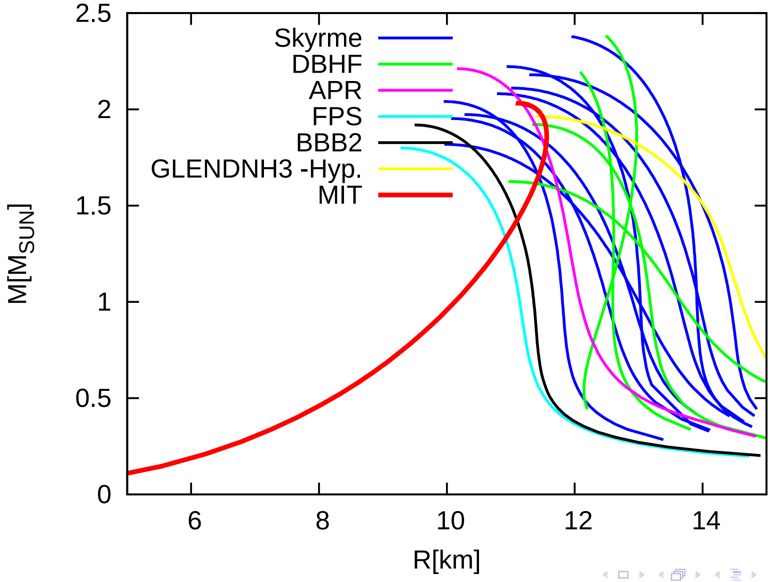
$$\frac{\mathrm{d}p_0}{\mathrm{d}r} = -(\rho_0 + p_0)\frac{m(r) + 4\pi r^3 p_0}{r(r - 2m(r))}$$

One integrate the TOV eq. for given ρ_c.
The relation between energy density and pressure $P(\rho)$ is needed (EOS).

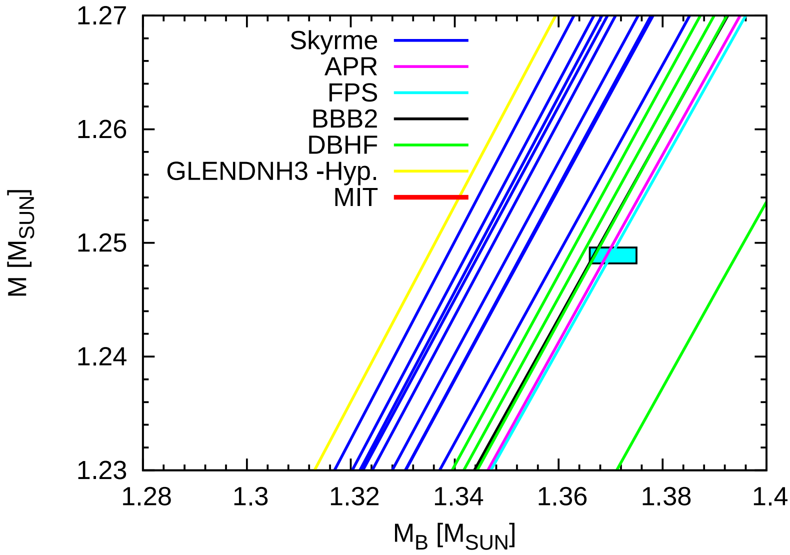
Compact stars mass

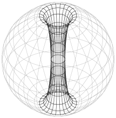


Mass - Radius relation



Mass - Baryonic mass relation





Hartle Thorne Approximation

- For given central parameters ($\rho_{\rm c}$, $P_{\rm c}$) one calculate the static nonrotating configuration. M_0
- For arbitrarily chosen value of $\tilde{\omega}_{\rm c}$ one calculate the angular velocity of fluid relative to velocity of local inertial frame $\tilde{\omega}(r) = \Omega - \omega(r)$ J
- Then one calculates the spherical deformations of the star to obtain total mass M
- The last thing one should do is to calculate the quadrupole deformations, which gives you the shape of the star. Q

Hartle Thorne perturbation

Hartle Thorne metric

$$
\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{e}^{2\nu_0}[1 + 2h_0(r) + 2h_2(r)P_2(\cos\theta)]\mathrm{d}t^2 \\
&+ \mathrm{e}^{2\lambda_0}\left\{1 + \frac{\mathrm{e}^{2\lambda_0}}{r}[2m_0(r) + 2m_2(r)P_2(\cos\theta)]\right\}\mathrm{d}r^2 \\
&+ r^2[1 + 2k_2(r)P_2(\cos\theta)]\left\{\mathrm{d}\theta^2 + [\mathrm{d}\phi - \omega(r)\mathrm{d}t]^2\sin^2\theta\right\}
\end{aligned}
$$

- $\omega(r)$ is order of Ω and describes the so-called dragging of the inertial frame
- $h_0(r), h_2(r), m_0(r), m_2(r), k_2(r)$, that are quantities of order Ω^2 and functions of radial coordinate r only
- All perturbation functions are calculated with appropriate boundary conditions at the center and at the surface of configuration.

Calculation of J

- One starts with integration of differential equation for $\tilde{\omega}(r) = \Omega - \omega(r)$:

$$\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j(r)\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right) + 4\frac{\mathrm{d}j}{\mathrm{d}r}\tilde{\omega} = 0$$

where $j = \mathrm{e}^{-(\lambda_0+\nu_0)}$ with boundary conditions $r = 0 : \ \tilde{\omega} = \tilde{\omega}_{\mathrm{c}}, \ \mathrm{d}\tilde{\omega}/\mathrm{d}r = 0$. Because we usually want to put the rotational frequency Ω as an input parameter we have to rescale the $\tilde{\omega}$ with respect to simple relation

$$\tilde{\omega}_{\mathrm{new}}(r) = \tilde{\omega}_{\mathrm{old}}(r)\frac{\Omega_{\mathrm{WE\ WANT}}}{\Omega_{\mathrm{WE\ GET}}}$$

- The angular momentum J and moment of inertia I are given by equations

$$J = \frac{1}{6}R^4\left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)_{r=R}, \quad I = \frac{J}{\Omega}$$

Calculation of M

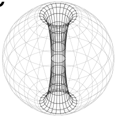
- To calculate the total mass of rotating neutron star one should solve set of two coupled differential equation one for m_0 and one for p_0

$$m_0(r): \quad \frac{\mathrm{d}m_0}{\mathrm{d}r} = 4\pi r^2(\rho + p)\frac{\mathrm{d}\rho}{\mathrm{d}p}\delta p_0 + \frac{1}{12}r^4 j^2\left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)^2 - \frac{1}{3}r^3\tilde{\omega}^2\frac{\mathrm{d}j^2}{\mathrm{d}r}$$

$$p_0(r): \quad \frac{\mathrm{d}p_0}{\mathrm{d}r} = -\frac{m_0(1+8\pi r^2 p)}{(r-2m)^2} - \frac{4\pi(\rho+p)r^2}{r-2m}p_0 + \frac{1}{12}\frac{r^4 j^2}{r-2m}\left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)^2 + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3 j^2\tilde{\omega}^2}{r-2m}\right)$$

- The boundary condition are that m_0 and h_0 must vanish at the center of the star. And at the surface they have to reach the external field
- The mass of rotational object is then given by

$$M(R) = M_0(R) + m_0(R) + J^2/R^3$$



Calculation of Q

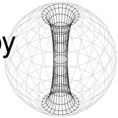
- To calculate the quadrupole of rotating neutron star one should solve set of two coupled differential equation one for $v_2 = h_2 + k_2$ and one for h_0

$$\frac{\mathrm{d}v_2}{\mathrm{d}r} = -2\frac{\mathrm{d}\nu_0}{\mathrm{d}r}h_2 + \left(\frac{1}{r} + \frac{\mathrm{d}\nu_0}{\mathrm{d}r}\right)\left[\frac{1}{6}r^4 j^2 \left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)^2 - \frac{1}{3}r^3\tilde{\omega}^2\frac{\mathrm{d}j^2}{\mathrm{d}r}\right]$$

$$\begin{aligned}\frac{\mathrm{d}h_2}{\mathrm{d}r} &= -\frac{2v_2}{r(r-2m(r))\mathrm{d}\nu_0/\mathrm{d}r} + \left\{-2\frac{\mathrm{d}\nu_0}{\mathrm{d}r} + \frac{r}{2(r-2m(r))\mathrm{d}\nu_0/\mathrm{d}r}\left[8\pi(\rho+p) - \frac{4m(r)}{r}\right]\right\}h_2 \\ &+ \frac{1}{6}\left[r\frac{\mathrm{d}\nu_0}{\mathrm{d}r} - \frac{1}{2(r-2m(r))\mathrm{d}\nu_0/\mathrm{d}r}\right]r^3 j^2\left(\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}r}\right)^2 - \frac{1}{3}\left[r\frac{\mathrm{d}\nu_0}{\mathrm{d}r} + \frac{1}{2(r-2m(r))\mathrm{d}\nu_0/\mathrm{d}r}\right]r^2\tilde{\omega}^2\frac{\mathrm{d}j^2}{\mathrm{d}r}\end{aligned}$$

The boundary condition are that v_2 and h_2 must vanish at the center of the star. And at the surface they have to reach the external field

The quadrupole moment of rotational object is then given by



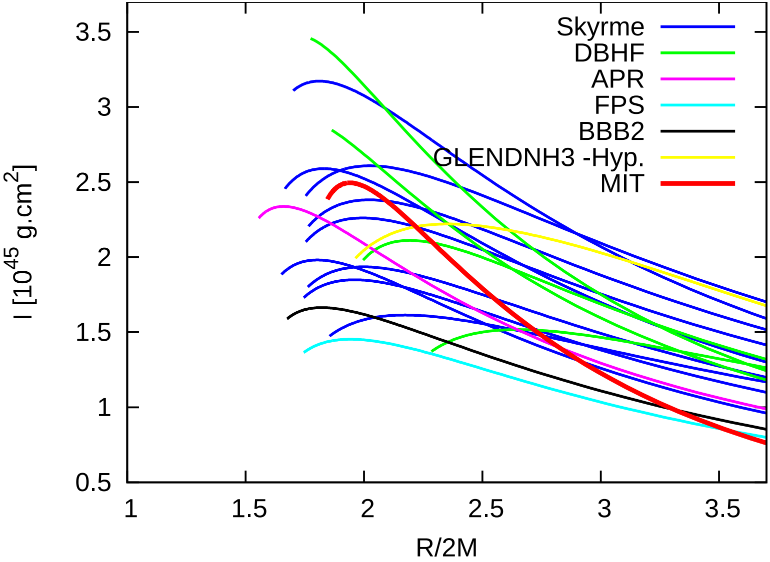
$$Q = 8/5KM^3 + J^2/M$$

Important quantities

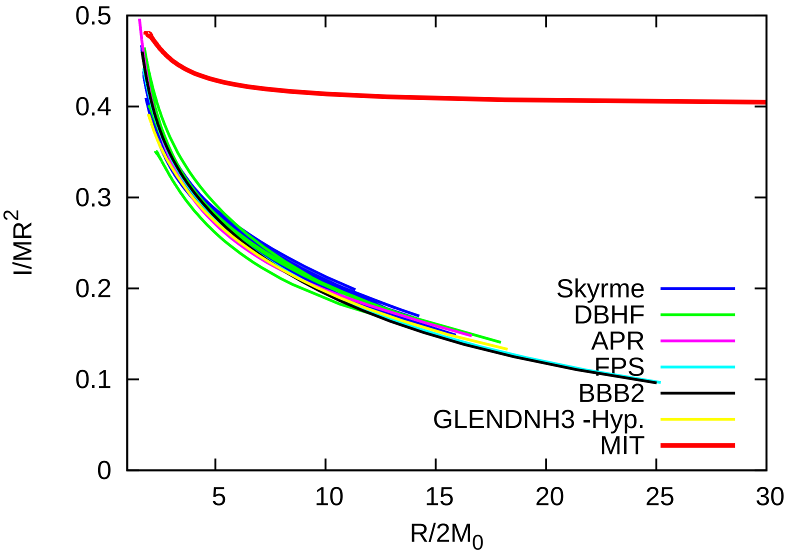
The useful quantities to calculate are are then

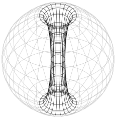
- Mass of the rotating compact star $M = M(\Omega^2)$
- Angular momentum of the rotating compact star $J = J(\Omega)$
- Moment of inertia of the rotating compact star $I = J/\Omega$
- Quadrupole moment of rotating compact star $Q = Q(\Omega^2)$
- "Kerr parameter" QM/J^2

Moment of inertia

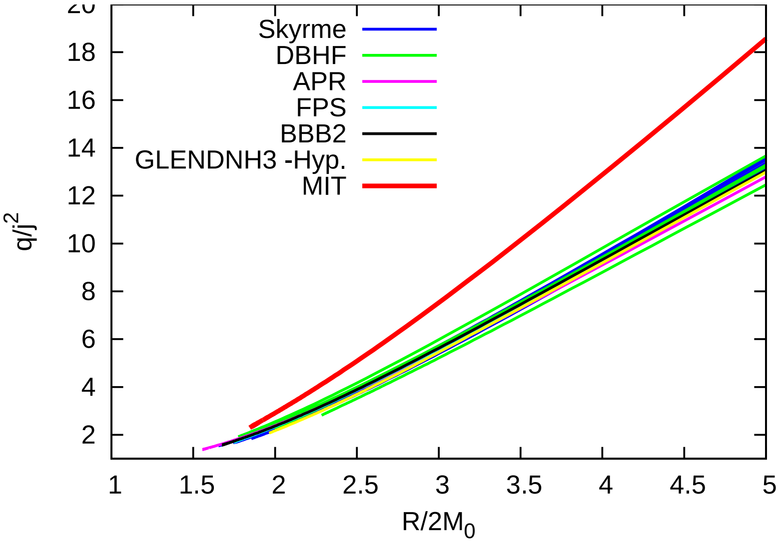


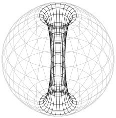
Moment of inertia





Kerr parameter

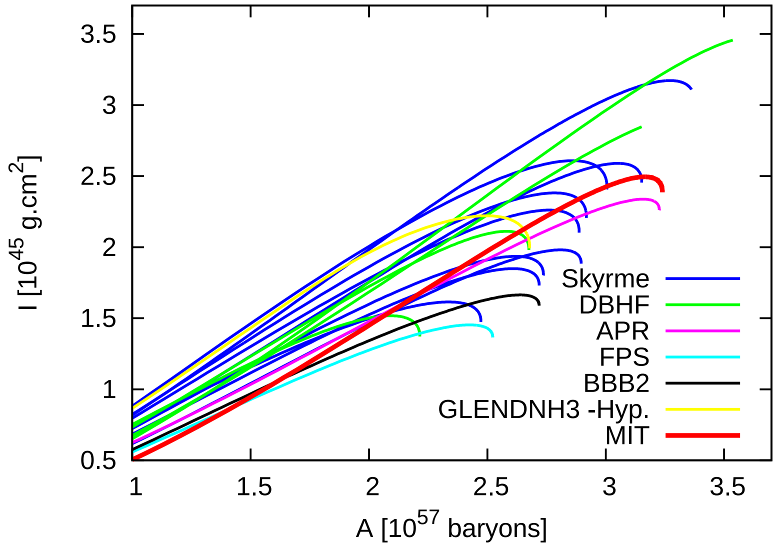




Conclusions

- We have tested number EOS to describe the properties of slowly rotating compact stars (NS and SS)
- We used the Hartle - Thorne approach, that describes the "slow rotation" approximation, that could be however used to describe the known objects.
- We have shown the differency between neutron and strange stars
- Future plan
  - Investigate rotational sequences of constant baryon number.
  - Compare slow rotating approximation with some public available codes for general rotation `(Lorene, RNS)`

Discussion - IA plot



Gabo's talk- $g_{\mu\nu}^{HT} / g_{\mu\nu}^{Kerr}$