

Black Holes Admitting Strong Resonant Phenomena

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ABSTRACT

High-frequency twin peak quasiperiodic oscillations (QPOs) are observed in four microquasars, *i.e.*, Galactic black hole binary systems, with frequency ratio very close to 3 : 2. In the microquasar GRS 1915+105, the structure of QPOs exhibits additional frequencies, and more than two frequencies are observed in the Galaxy nuclei Sgr A*, or in some extragalactic sources (NGC 4051, MCG-6-30-15 and NGC 5408 X-1). The observed QPOs can be explained by a variety of the orbital resonance model versions assuming resonance of oscillations with the Keplerian frequency ν_K or the vertical epicyclic frequency ν_θ , and the radial epicyclic frequency ν_r , or some combinations of these frequencies. Generally, different resonances could arise at different radii of an accretion disc. However, we have shown that for special values of dimensionless black hole spin a strong resonant phenomena could occur when different resonances can be excited at the same radius, as cooperative phenomena between the resonances may work in such situations. The special values of a are determined for triple frequency ratio sets $\nu_K : \nu_\theta : \nu_r = s : t : u$ with s, t, u being small integers. The most promising example of such a special situation arises for black holes with extraordinary resonant spin $a = 0.983$ at the radius $r = 2.395M$, where $\nu_K : \nu_\theta : \nu_r = 3 : 2 : 1$. We also predict that when combinations of the orbital frequencies are allowed, QPOs with four frequency ratio set 4 : 3 : 2 : 1 could be observed in the field of black holes with $a = 0.866, 0.882$ and 0.962 . Assuming the extraordinary resonant spin $a = 0.983$ in Sgr A*, its QPOs with observed frequency ratio $\approx 3 : 2 : 1$ imply the black hole mass in the interval $4.3 \times 10^6 M_\odot < M < 5.4 \times 10^6 M_\odot$, in agreement with estimates given by other, independent, observations.

Key words: *Accretion, accretion disks – Black hole physics – X-rays: general*

1 Introduction

Quasiperiodic oscillations (QPOs) of X-ray brightness had been observed in many Galactic low-mass X-ray binaries containing neutron stars (see, *e.g.*, van der Klis 2000, 2006, Barret *et al.* 2005, Belloni *et al.* 2005, 2007) or black holes (see, *e.g.*, McClintock and Remillard 2004, Remillard 2005, Remillard and McClintock 2006). Some of the QPOs are in the kHz range and often come in pairs ($\nu_{\text{upp}}, \nu_{\text{down}}$) of *twin peaks* (often called *double peaks*) in the Fourier power spectra. Since the peaks of high frequencies are close to the orbital frequency of the marginally stable circular orbit representing the inner edge of Keplerian discs orbiting black holes (or neutron stars), the strong gravity effects have to be relevant in explaining high-frequency QPOs (Abramowicz *et al.* 2004).

The twin peak QPOs were observed in four microquasars, namely GRO 1655-40, XTE 1550-564, H 1743-322, GRS 1915+105 (Török *et al.* 2005). In all of the four cases, the frequency ratio of the twin peaks is very close to 3 : 2 and the orbital resonance model assuming non-linear resonances between oscillations in Keplerian ν_K and epicyclic frequencies ν_θ (vertical), or ν_r (radial), or their combinations, seems to be the potential explanation of the microquasar kHz QPOs. The resonances of oscillations with the epicyclic frequencies were firstly mentioned in Aliev and Galtsov (1981). In the context of QPOs observed in the neutron star and black hole systems, the orbital resonance model was introduced and developed by Kluźniak and Abramowicz (2000), see Abramowicz *et al.* (2004).

According to the resonance hypothesis, the two modes in resonance should have eigenfrequencies ν_r (equal to the radial epicyclic frequency) and ν_ν (equal to the vertical epicyclic frequency ν_θ or to the Keplerian frequency ν_K). While models based

on the *parametric resonance* identify the two observed frequencies of the twin peaks (ν_{upp} , ν_{down}) directly with the eigenfrequencies of a resonance, models based on the *forced resonance* allow to observe combinational (beat) frequencies of the modes, or oscillations with the combinational frequencies entering the resonance (Landau and Lifshitz 1976). Both parametric and forced resonance models make clear and precise predictions about the values of observed frequencies in connection with spin and mass of the observed object, at least in the case of black holes (Török *et al.* 2005).

In some sources, more than two high-frequency peaks are observed. The microquasar GRS 1915+105 reveals high-frequency QPOs appearing at four frequencies with the lower and upper pairs in the ratio close to 3 : 2 (Remillard and McClintock 2006) and even a fifth frequency was reported (Belloni *et al.* 2001). An additional sixth frequency was mentioned (Strohmayer 2001), although not confirmed. In Sgr A*, three frequencies were reported with ratio close to 3 : 2 : 1 (Aschenbach 2004, Aschenbach *et al.* 2004, Török 2005). In the galactic nuclei MCG-6-30-15 and NGC 4051, two pairs of QPOs were reported with the ratios close to 3 : 2 and 2 : 1, respectively (Lachowicz *et al.* 2006). In the source NGC 5408 X-1 oscillations with three frequencies of ratio close to 6 : 4 : 3 were observed (Strohmayer *et al.* 2007).

In the microquasar GRS1915+105, an almost extreme Kerr black hole with $a \approx 1$ is expected (McClintock *et al.* 2006), and all the five (six) frequencies of QPOs can be explained in the framework of the extended resonance model with the hump-induced oscillations, predicting the black hole spin $a = 0.9998$ and its mass $M = 14.8 M_{\odot}$ (Stuchlík *et al.* 2006, 2007d). In the extended resonance model, forced resonances of the epicyclic oscillations with an additional oscillation induced by the humpy orbital velocity profile (related to the physically privileged locally non-rotating frames) that occurs in Keplerian discs orbiting Kerr black holes with $a > 0.9953$ are assumed (Aschenbach 2004, 2006, Stuchlík *et al.* 2004, 2005, 2007c). In the “humpy” extended resonance model, all the oscillations in resonance can occur at, and be related to the exclusively defined, “humpy radius” with extremal orbital velocity gradient within the humpy profile (Stuchlík *et al.* 2007c). However, this model can be relevant only for near-extreme Kerr black holes with spin $a > 0.9953$ for Keplerian discs, and with even higher spin $a > 0.9998$ for marginally stable thick accretion discs (Stuchlík *et al.* 2004, 2005).

In order to explain complex frequency structures observed in some black hole systems, we expect that more than one resonance occur in a Keplerian disc, and different versions of the orbital resonance model could be acting. Of course, one version of the orbital resonance model appearing at different resonant points, at different radii of the disc, is also possible (Stuchlík and Török 2005). Note that the so called total precession resonance model, assuming resonance of oscillations with the Keplerian frequency ν_K and the total precession frequency $\nu_T = \nu_{\theta} - \nu_r$, can explain quite well the multi-resonant phenomena observed in the neutron star atoll source 4U 1636-53 (Stuchlík *et al.* 2007e) and in some other atoll sources (Bakala *et al.* 2008, in preparation).

Generally, two resonances could occur at different radii, and for special values of the dimensionless spin a , the bottom, top, or mixed frequencies coincide, reducing two frequency pairs into a triple frequency set (Stuchlík *et al.* 2007b). Physically more interesting seems to be the case, when a triple frequency set $\nu_K, \nu_{\theta}, \nu_r$ with rational ratio arises at a given radius. The resonant phenomena can be then expected stronger than in the case of internally independent resonances occurring at two different radii. Clearly, while sharing a given radius, the resonances could be causally related and could cooperate efficiently (Landau and Lifshitz 1976). That is the reason why we focus here attention on the possibility of causally related resonances appearing at a fixed radius. Such a situation is possible only in the field of Kerr black holes with special values of the dimensionless spin a , when the value of a is related to the corresponding triple fre-

quency ratio set, and concrete versions of the resonance model that are realized. Since the strength of the resonance and the resonant frequency width decrease rapidly with the order of the resonance $n + m$ (see Landau and Lifshitz 1976), in the following we restrict ourselves to the cases with $n, m \leq 5$.

2 Orbital Resonance Model

The standard orbital resonance model (Abramowicz *et al.* 2004, Török *et al.* 2005) assumes oscillations of an accretion disc orbiting a rotating black hole described by the Kerr geometry, or a neutron star that could be represented by the Schwarzschild geometry, or, more precisely, by the Hartle–Thorne geometry (Hartle and Thorne 1968). The accretion disc can be approximated by a thin disc with Keplerian angular velocity profile, or by a thick toroidal disc with angular velocity profile given by distribution of the specific angular momentum in the fluid of the toroidal disc. The frequency of the disc oscillations is related to the Keplerian frequency (orbital frequency of tori), or the radial and vertical epicyclic frequencies of the circular test particle (geodetical) motion. The epicyclic frequencies can be relevant both for the thin, Keplerian discs with quasicircular geodetical motion (Kato *et al.* 1998, Novikov and Thorne 1973) and for thick, toroidal discs with non-geodetical quasicircular rotation kept by the pressure gradients of the tori (Schnittman and Rezzolla 2006, Rezzolla *et al.* 2003, Rezzolla 2004ab). However, with thickness of an oscillating toroid growing, the eigenfrequencies of its radial and vertical oscillations deviate from the epicyclic test particle frequencies (Šrámková 2005). Here we focus our attention to the Keplerian thin discs.

Different versions of the orbital resonance model could be classified according to the following criteria:

- a) the type of the resonance (parametric or forced),
- b) the type of oscillations entering the resonance,
- c) the presence of beat, combinational frequencies.

Thus, according to the first criterion, two main groups of orbital resonance model versions exist, differing by the type of the resonance. In both of them, the epicyclic frequencies of the equatorial test particle circular motion have a crucial role (Török *et al.* 2005).

The internal resonance model is based on the idea of *parametric resonance* between vertical and radial epicyclic oscillations with the frequencies $\nu_\theta = \omega_\theta/2\pi$ and $\nu_r = \omega_r/2\pi$. The parametric resonance is described by the Mathieu equation (Landau and Lifshitz 1976)

$$\delta\ddot{\theta} + \omega_\theta^2 [1 + h \cos(\omega_r t)] \delta\theta = 0. \quad (1)$$

Theory behind the Mathieu equation implies that a parametric resonance is excited when

$$\frac{\omega_r}{\omega_\theta} = \frac{\nu_r}{\nu_\theta} = \frac{2}{n}, \quad n = 1, 2, 3, \dots \quad (2)$$

and is strongest for the smallest possible value of n (Landau and Lifshitz 1976). Because $\nu_r < \nu_\theta$ near black holes, the smallest possible value for the parametric resonance is $n = 3$, which means that $2\nu_\theta = 3\nu_r$. This explains why the 3 : 2 ratio is commonly observed in the black hole systems, assuming $\nu_{\text{upp}} = \nu_\theta$ and $\nu_{\text{down}} = \nu_r$. Note that for the internal resonance the oscillating system conserves energy (Landau and Lifshitz 1976).

Versions of the resonance model based on the *forced resonance* come from the idea of a forced non-linear oscillator, when the relation of the latitudinal (vertical) and radial oscillations is given by the formulae

$$\delta\ddot{\theta} + \omega_{\theta}^2\delta\theta + [\text{non-linear terms in } \delta\theta] = g(r)\cos(\omega_0 t), \quad (3)$$

$$\delta\ddot{r} + \omega_r^2\delta r + [\text{non-linear terms in } \delta\theta, \delta r] = h(r)\cos(\omega_0 t), \quad (4)$$

with

$$\omega_{\theta} = \left(\frac{p}{q}\right)\omega_r, \quad (5)$$

where p, q are small natural numbers and ω_0 is the frequency of the external force, *e.g.*, the gravitational perturbative forces are discussed, for the case of a neutron star with “mountains” or accretion columns, and a binary partner of the neutron star or a black hole, in Stuchlík *et al.* (2008, 2007a), Stuchlík and Hledík (2005). The non-linear terms allow the presence of combination (beat) frequencies in resonant solutions for $\delta\theta(t)$ and $\delta r(t)$ (see, *e.g.*, Landau and Lifshitz 1976), which in the simplest case give

$$\omega_{-} = \omega_{\theta} - \omega_r, \quad \omega_{+} = \omega_{\theta} + \omega_r. \quad (6)$$

Another, so called “*Keplerian*” resonance model, takes into account possible parametric or forced resonances between oscillations with the radial epicyclic frequency ν_r or the vertical epicyclic frequency ν_{θ} , and the Keplerian orbital frequency ν_K .

Such resonances can produce the observable frequencies in the 3 : 2 ratio as well as in other rational ratios (note that one of the cases which gives 3 : 2 observed ratio is also the “direct” case of $p : q = 3 : 2$ corresponding to the same frequencies and radius as in the case of 3 : 2 parametric resonance). Therefore, we shall consider both the direct and simple combinational resonances.

The resonance conditions of the parametric and direct forced resonance are common, however, physical details, as the time evolution of the resonance, the strength of the resonance and the width of the resonant frequencies, are different (see Landau and Lifshitz 1976).

The width of the resonant frequencies in forced resonances differs in the cases of direct, sub(super)harmonic and combinational resonances, and depends on the external force strength and the damping and non-linear terms. Considering a simple case of external force of harmonic character with frequency Ω influencing a non-linear oscillator with a single degree of freedom and variable u , having eigenfrequency ω_0 , damping parameter μ and cubic non-linear term characterized by parameter α , the equation of motion reads (Nayfeh and Mook 1979)

$$\ddot{u} + \omega_0^2 u = -2\varepsilon\mu\dot{u} - \varepsilon\alpha u^3 + \varepsilon k \cos(\Omega t) \quad (7)$$

where ε is a small parameter. For primary resonance with $\Omega \sim \omega_0$ in the linear regime, where both damping and non-linearities are negligible, the amplitude growing is unbounded according to $A \sim t$, but $\Omega = \omega_0$ exactly (Landau and Lifshitz 1976). The linear growing of amplitude is limited by damping and non-linearities and detuning of the frequencies is allowed. We describe the frequency detuning by a detuning parameter σ due to

$$\Omega = \omega_0 + \varepsilon\sigma. \quad (8)$$

The frequency-response equation, relating the amplitude of the resulting oscillations A and σ in dependence on the amplitude of the excitation k , and the parameters μ and α , can be put into the form (Nayfeh and Mook 1979)

$$\sigma = \frac{3\alpha}{8\omega_0} A^2 \pm \left(\frac{k^2}{4\omega_0^2 A^2} - \mu^2 \right)^{1/2}. \quad (9)$$

The linear response ($\alpha = 0$) is symmetric, while $\alpha > 0$ introduces an asymmetry – in both cases the peak amplitude

$$A_{\max} = \frac{k}{2\omega_0\mu}; \quad (10)$$

(see Nayfeh and Mook 1979 for details). The maximal frequency detuning allowed for resonant phenomena represents few percent of the eigenfrequency ω_0 .

In the case of superharmonic resonances

$$3\Omega = \omega_0 + \varepsilon\sigma, \quad (11)$$

the frequency-response equation reads

$$\sigma = 3\frac{\alpha\Lambda^2}{\omega_0} + \frac{3\alpha}{8\omega_0}A^2 \pm \left(\frac{\alpha^2\Lambda^6}{\omega_0^2A^2} - \mu^2 \right)^{1/2}, \quad (12)$$

where

$$\Lambda = \frac{k}{2(\omega_0^2 - \Omega^2)}. \quad (13)$$

The peak amplitude of the resonant overtone oscillation is

$$A_{\max} = \frac{\alpha\Lambda^3}{\omega_0\mu}, \quad (14)$$

i.e., it depends on the magnitude of the non-linear term, contrary to the case of primary resonances. Similar frequency-response equations can be given for the subharmonic resonance ($\Omega = 3\omega_0 + \varepsilon\sigma$) or for combinational resonances (Nayfeh and Mook 1979). Generally, the maximal frequency scatter represents few percent of ω_0 again (Nayfeh and Mook 1979). Similarly, in the case of the quadratic non-linearity ($-\varepsilon\alpha u^2$) introducing the sub(super)harmonic frequencies $\Omega \sim 2\omega_0$ ($\Omega \sim \frac{1}{2}\omega_0$).

In the case of the parametric resonance, where the excitation appears as a time-dependent coefficient in oscillatory equations, and for internal resonances between coupled oscillatory modes, the frequency detuning is given in a more complex way (Nayfeh and Mook 1979, Landau and Lifshitz 1976) but the maximal relative frequency detuning is again in percents and sharply decreases with the order of the resonance (Landau and Lifshitz 1976). We shall not go into details leaving them to future work. Here we concentrate on the resonant conditions.

The formulae for the vertical epicyclic frequency ν_θ and the radial epicyclic frequency ν_r take in the gravitational field of a rotating Kerr black hole (with the mass M and dimensionless spin a) the form (*e.g.*, Aliev and Galtsov 1981, Kato *et al.* 1998)

$$\nu_\theta^2 = \alpha_\theta \nu_K^2, \quad \nu_r^2 = \alpha_r \nu_K^2 \quad (15)$$

where the Keplerian frequency and related dimensionless epicyclic frequencies are given by the formulae

$$\begin{aligned} \nu_K &= \frac{1}{2\pi} \left(\frac{GM}{r_G^3} \right)^{1/2} (x^{3/2} + a)^{-1} = \frac{1}{2\pi} \left(\frac{c^3}{GM} \right) (x^{3/2} + a)^{-1}, \\ \alpha_\theta &= 1 - 4ax^{-3/2} + 3a^2x^{-2}, \\ \alpha_r &= 1 - 6x^{-1} + 8ax^{-3/2} - 3a^2x^{-2}. \end{aligned} \quad (16)$$

Here $x = r/(GM/c^2)$ is the dimensionless radius, expressed in terms of the gravitational radius of the black hole.

For a particular resonance $n : m$ the equation

$$n\nu_r = m\nu_v; \quad \nu_v \in \{\nu_\theta, \nu_K\} \quad (17)$$

determines the dimensionless resonance radius $x_{n:m}$ as a function of the dimensionless spin a in the case of direct resonances that can be easily extended to the resonances with combinational frequencies. From the known mass of the central black hole (*e.g.*, low-mass in the case of binary systems or high-mass in the case of supermassive black holes), and the observed twin peak frequencies (ν_{upp} , ν_{down}), the Eqs (15)–(17) imply the black hole spin a for various versions of the resonance model, with the beat frequencies taken into account (Török *et al.* 2005).

3 Black Holes Allowing Multiple Resonances

Generally, the resonances could be excited at different radii of the accretion disc under different internal conditions; such a situation is discussed in detail by Stuchlík *et al.* (2007b). On the other hand, physically more interesting case arises when two (or more) resonances are excited at a common radius because of probable internal physical connection of those resonances. Of course, a causal connection of two resonances could hardly be relevant, but is not excluded, if they appear at distant radii.

A simple possibility of two resonances sharing a common radius can be conceived supposing a direct resonance and some combinational resonances of the frequencies entering the direct resonances. For example, in the field of any Kerr black hole a direct resonance of the epicyclic frequencies $\nu_\theta : \nu_r = 3 : 2$ admits combinational resonances $\nu_\theta : (\nu_\theta - \nu_r) = 3 : 1$ and $\nu_r : (\nu_\theta - \nu_r) = 2 : 1$ (see, *e.g.*, Stuchlík *et al.* 2007b). However, this kind of behaviour is not exhibited by any of the four microquasars investigated at present time (Török *et al.* 2005).

Nevertheless, it is important that there exists a possibility of direct resonances of oscillations with all of the three orbital frequencies, characterized by a triple frequency ratio set

$$\nu_K : \nu_\theta : \nu_r = s : t : u \quad (18)$$

with $s > t > u$ being small integers. The frequency set ratio (Eq. 18) can be realized only for special values of the black hole spin a . The black hole mass is then related to the magnitude of the frequencies.

In order to look for the special values of the black hole spin a and the related frequency ratios (Eq. 18), we have to summarize properties of the Keplerian and epicyclic frequencies of the circular geodesic motion in the Kerr spacetimes. It is well known that ν_K and ν_θ are defined down to the photon circular geodesic located at x_{ph} that is determined by the condition

$$a = a_{\text{ph}}(x) \equiv \frac{\sqrt{x}}{2}(3 - x) \quad (19)$$

while ν_r is defined down to the innermost (co-rotating) stable circular geodesic at x_{ms} given by

$$a = a_{\text{ms}}(x) \equiv \frac{\sqrt{x}}{3} \left(4 - \sqrt{3x - 2} \right). \quad (20)$$

Clearly, we have to search for the integer ratios (Eq. 18) at $x \geq x_{\text{ms}}$.

The Keplerian frequency $\nu_K(r, a)$ is a monotonically decreasing function of the radial coordinate for any value of the black hole spin. On the other hand, the radial epicyclic frequency has the global maximum for any Kerr black hole. The vertical epicyclic frequency is not monotonic, if the spin is sufficiently high (see, *e.g.*, Kato

et al. 1998, Perez *et al.* 1997). For the Kerr black hole spacetimes, the locations $\mathcal{R}_r(a)$, $\mathcal{R}_\theta(a)$ of maxima of the epicyclic frequencies ν_r , ν_θ are implicitly given by the conditions (Török and Stuchlík 2005a)

$$\begin{aligned}\beta_j(x, a) &= \frac{1}{2} \frac{\sqrt{x}}{x^{3/2} + a} \alpha_j(x, a) \quad \text{where } j \in \{r, \theta\}, \\ \beta_r(x, a) &\equiv \frac{1}{x^2} - \frac{2a}{x^{5/2}} + \frac{a^2}{x^3}, \\ \beta_\theta(x, a) &\equiv \frac{a}{x^{5/2}} - \frac{a^2}{x^3}.\end{aligned}\tag{21}$$

For any black hole spin, the extrema of the radial epicyclic frequency $\mathcal{R}_r(a)$ must be located above the marginally stable orbit. On the other hand, the latitudinal extrema $\mathcal{R}_\theta(a)$ are located above the photon (marginally bound or marginally stable) circular orbit only if the limits on the black hole spin $a > 0.748$ (0.852, 0.952) are satisfied (Török and Stuchlík 2005b). In the Keplerian discs, with the inner boundary $x_{\text{in}} \sim x_{\text{ms}}$, the limiting value $a = 0.952$ is relevant.

From the point of view of the observational consequences, it is important to know, for which frequency ratios $n:m$ the resonant frequencies $\nu_\theta(a, n:m)$, considered as a function of the black hole spin a for a given frequency ratio $n:m$, has a non-monotonic character. A detailed analysis (Török and Stuchlík 2005a) shows that $\nu_\theta(a, n:m)$ has a local maximum for $n:m > 11:5$, *i.e.*, in physically relevant situations (n, m small enough for the resonance), it occurs for the ratios $\nu_\theta : \nu_r = 3:1, 4:1, 5:2, 5:1$.

Assuming two resonances $\nu_K : \nu_\theta = s:t$ and $\nu_K : \nu_r = s:u$ occurring at the same x , we arrive to the conditions

$$\alpha_\theta(a, x) = \left(\frac{t}{s}\right)^2, \tag{22}$$

$$\alpha_r(a, x) = \left(\frac{u}{s}\right)^2 \tag{23}$$

that have to be solved simultaneously for x and a . The solution is given by the condition

$$a^\theta(x, t/s) = a^r(x, u/s) \tag{24}$$

where

$$a^\theta(x, t/s) \equiv \frac{\sqrt{x}}{3} \left(2 \pm \sqrt{4 - 3x \left[1 - \left(\frac{t}{s}\right)^2 \right]} \right), \tag{25}$$

$$a^r(x, u/s) \equiv \frac{\sqrt{x}}{3} \left(4 \pm \sqrt{-2 + 3x \left[1 - \left(\frac{u}{s}\right)^2 \right]} \right). \tag{26}$$

It is possible to find an explicit solution determining the relevant radius for any triple frequency set ratio $s:t:u$

$$x(s, t, u) = \frac{6s^2}{6s^2 \pm 2\sqrt{2}\sqrt{(t-u)(t+u)}(3s^2 - t^2 - 2u^2) - (t^2 + 5u^2)}. \tag{27}$$

Clearly, the condition $t^2 + 2u^2 \leq 3s^2$ is always satisfied. The corresponding black hole spin a is then determined, *e.g.*, by Eq. (25) giving $a^\theta(x(s, t, u), t/s)$. Of course, we consider only the black hole cases when $a \leq 1$. This condition puts a restriction on allowed values of s, t, u .

The solutions have been found for frequency ratios with $s \leq 5$. We were looking for the black hole spin a allowing ratios $s:t:u = 3:2:1, 4:3:2, 4:3:1, 4:2:1, 5:4:3$,

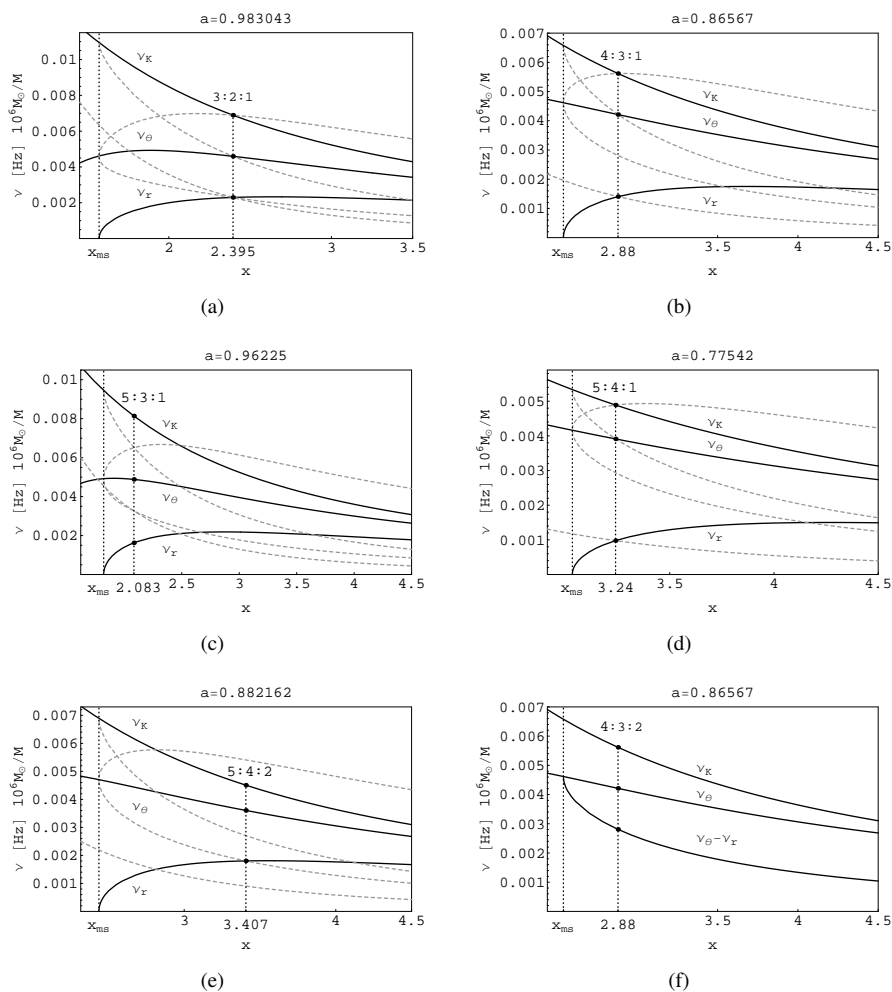


Fig. 1. Black holes enabling interrelated strong resonances. The special cases of the triple frequency ratio sets of the orbital frequencies ν_K , ν_θ , ν_r (black solid lines) with the corresponding spin a and the shared resonance radius. For completeness we present the relevant simple combinational frequencies $\nu_\theta - \nu_r$, $\nu_\theta + \nu_r$, $\nu_K - \nu_\theta$, $\nu_K + \nu_r$ (grey dashed lines). Notice that the extraordinary resonant spin $a_{3,2:1} = 0.983$ represents the only case when the combinational and direct orbital frequencies coincide at the shared resonance radius.

5:4:2, 5:4:1, 5:3:2, 5:3:1, 5:2:1. Higher values of the s, t, u are presented and discussed in a more general context of triple frequency set ratios that could arise when two resonances occur at two different radii, but for special values of the black hole spin two of the frequency levels (top, bottom, mixed) coincide (see Stuchlík *et al.* 2007b).

We have shown that the direct resonances could result only in the triple frequency sets 3:2:1, 4:3:1, 5:4:2, 5:4:1, 5:3:1 with values of the spin and the radius $x_{s:t:u}$ given in Fig. 1a–e. The other ratios can be realized with the combinational frequencies involved in the triple frequency sets 4:3:2, 4:2:1, 5:4:3, 5:3:2, 5:2:1 as shown in Fig. 1f for the case of the set 4:3:2. We can summarize the results in the following way.

a) $\nu_K : \nu_\theta : \nu_r = 3 : 2 : 1$ (see Fig. 1a)

$$a_{3:2:1} = 0.983, \quad x_{3:2:1} = 2.395, \quad x_{ms} = 1.571. \quad (28)$$

This case of the so called extraordinary resonant spin $a_{3:2:1} = 0.983$ may allow strong resonances, in situations when the Keplerian and epicyclic frequencies are

in the lowest possible ratio at the common radius $x_{3:2:1} = 2.395$ (see Fig. 1a), and is thus of special interest. In fact, this case involves the lowest structure of direct resonances with $\nu_K : \nu_r = 3 : 1$, $\nu_K : \nu_\theta = 3 : 2$, $\nu_\theta : \nu_r = 2 : 1$. Notice that in this special case also any of the simple combinational frequencies coincides with one of the frequencies ν_K , ν_θ , ν_r , and are in the fixed small integer ratios

$$\frac{\nu_K}{\nu_\theta - \nu_r} = \frac{3}{1}, \quad \frac{\nu_K}{\nu_K - \nu_r} = \frac{3}{2}, \quad \frac{\nu_\theta}{\nu_\theta - \nu_r} = \frac{2}{1}. \quad (29)$$

It should be stressed that this is the only case when the combinational frequencies not exceeding the Keplerian frequency are in the same ratios as the orbital frequencies. We obtain the strongest possible resonances when the beat frequencies enter the resonances satisfying the conditions

$$\frac{\nu_\theta + \nu_r}{\nu_K} = \frac{3}{3} = 1, \quad \frac{\nu_\theta}{\nu_K - \nu_r} = \frac{2}{2} = 1, \quad \frac{\nu_r}{\nu_K - \nu_\theta} = 1, \quad \frac{\nu_\theta - \nu_r}{\nu_r} = 1. \quad (30)$$

b) $\nu_K : \nu_\theta : \nu_r = 4 : 3 : 1$ (see Fig. 1b)

$$a_{4:3:1} = 0.866, \quad x_{4:3:1} = 2.88, \quad x_{\text{ms}} = 2.539. \quad (31)$$

In this case the combinational frequencies give additional frequency ratios

$$\frac{\nu_K}{\nu_K - \nu_r} = \frac{4}{3}, \quad \frac{\nu_\theta}{\nu_\theta - \nu_r} = \frac{3}{2}, \quad \frac{\nu_K}{\nu_\theta - \nu_r} = \frac{4}{2} = \frac{2}{1}, \quad \frac{\nu_\theta - \nu_r}{\nu_r} = \frac{2}{1}, \quad (32)$$

and the strongest resonant ratios

$$\frac{\nu_\theta + \nu_r}{\nu_K} = \frac{4}{4} = 1, \quad \frac{\nu_\theta}{\nu_K - \nu_r} = \frac{3}{3} = 1, \quad \frac{\nu_r}{\nu_K - \nu_\theta} = 1. \quad (33)$$

Here, using the combinational frequency $\nu_\theta - \nu_r$, we obtain the other three frequency ratio sets (see Fig. 1f)

$$\nu_K : \nu_\theta : (\nu_\theta - \nu_r) = 4 : 3 : 2 \quad (34)$$

and

$$\nu_K : (\nu_\theta - \nu_r) : \nu_r = 4 : 2 : 1. \quad (35)$$

Now, the four observable frequency ratio set is possible when the combinational frequency is mixed with the orbital frequencies

$$(\nu_K = \nu_\theta + \nu_r) : (\nu_\theta = \nu_K - \nu_r) : (\nu_\theta - \nu_r) : (\nu_r = \nu_K - \nu_\theta) = 4 : 3 : 2 : 1. \quad (36)$$

c) $\nu_K : \nu_\theta : \nu_r = 5 : 3 : 1$ (see Fig. 1c)

$$a_{5:3:1} = 0.962, \quad x_{5:3:1} = 2.083, \quad x_{\text{ms}} = 1.820, \quad (37)$$

and the combinational frequencies (not exceeding ν_K) are in the ratios

$$\begin{aligned} \frac{\nu_K}{\nu_K - \nu_r} &= \frac{\nu_K}{\nu_\theta + \nu_r} = \frac{5}{4}, & \frac{\nu_K - \nu_r}{\nu_\theta} &= \frac{4}{3}, & \frac{\nu_\theta}{\nu_\theta - \nu_r} &= \frac{3}{2}, \\ \frac{\nu_K - \nu_\theta}{\nu_r} &= \frac{\nu_\theta - \nu_r}{\nu_r} = \frac{2}{1}, & \frac{\nu_K}{\nu_\theta - \nu_r} &= \frac{5}{2}. \end{aligned} \quad (38)$$

Then we can generate triple frequency sets involving the combinational frequencies with the ratios in the form

$$\nu_K : (\nu_\theta - \nu_r) : \nu_r = 5 : 2 : 1 \quad (39)$$

and

$$\nu_K : \nu_\theta : (\nu_\theta - \nu_r) = 5 : 3 : 2. \quad (40)$$

Moreover, using the combinational frequencies we could obtain two sets of four frequency ratios

$$\begin{aligned} \nu_K : (\nu_K - \nu_r = \nu_\theta + \nu_r) : \nu_\theta : \nu_r &= 5 : 4 : 3 : 1, \\ \nu_K : \nu_\theta : (\nu_\theta - \nu_r = \nu_K - \nu_\theta) : \nu_r &= 5 : 3 : 2 : 1, \end{aligned} \quad (41)$$

and one set of five frequency ratio

$$\nu_K : (\nu_K - \nu_r = \nu_\theta + \nu_r) : \nu_\theta : (\nu_\theta - \nu_r = \nu_K - \nu_\theta) : \nu_r = 5 : 4 : 3 : 2 : 1. \quad (42)$$

d) $\nu_K : \nu_\theta : \nu_r = 5 : 4 : 1$ (see Fig. 1d)

$$a_{5:4:1} = 0.775, \quad x_{5:4:1} = 3.240, \quad x_{ms} = 3.033. \quad (43)$$

The combinational frequencies can give the ratios

$$\frac{\nu_K}{\nu_K - \nu_r} = \frac{5}{4}, \quad \frac{\nu_\theta}{\nu_\theta - \nu_r} = \frac{4}{3}, \quad \frac{\nu_K}{\nu_\theta - \nu_r} = \frac{5}{3}, \quad \frac{\nu_\theta - \nu_r}{\nu_r} = \frac{3}{1}, \quad (44)$$

and the strongest resonant ratios

$$\frac{\nu_\theta + \nu_r}{\nu_K} = \frac{5}{5} = 1, \quad \frac{\nu_\theta}{\nu_K - \nu_r} = \frac{4}{4} = 1, \quad \frac{\nu_r}{\nu_K - \nu_\theta} = 1. \quad (45)$$

This case leads to the triple frequency ratio sets

$$\begin{aligned} \nu_K : (\nu_\theta - \nu_r) : \nu_r &= 5 : 3 : 1, \\ \nu_K : \nu_\theta : (\nu_\theta - \nu_r) &= 5 : 4 : 3. \end{aligned} \quad (46)$$

Here, only one of the four frequency ratio sets is possible, namely

$$(\nu_K = \nu_\theta + \nu_r) : (\nu_\theta = \nu_K - \nu_r) : (\nu_\theta - \nu_r) : \nu_r = 5 : 4 : 3 : 1. \quad (47)$$

e) $\nu_K : \nu_\theta : \nu_r = 5 : 4 : 2$ (see Fig. 1e)

$$a_{5:4:2} = 0.882, \quad x_{5:4:2} = 3.407, \quad x_{ms} = 2.438. \quad (48)$$

Here, the combinational frequencies give the ratios

$$\begin{aligned} \frac{\nu_\theta}{\nu_K - \nu_r} &= \frac{4}{3}, & \frac{\nu_K}{\nu_K - \nu_r} &= \frac{5}{3}, & \frac{\nu_\theta}{\nu_\theta - \nu_r} &= \frac{4}{2} = \frac{2}{1}, \\ \frac{\nu_r}{\nu_K - \nu_\theta} &= \frac{2}{1}, & \frac{\nu_K}{\nu_\theta - \nu_r} &= \frac{5}{2}, & & \end{aligned} \quad (49)$$

and the strongest resonant ratio

$$\frac{\nu_\theta - \nu_r}{\nu_r} = \frac{2}{2} = 1. \quad (50)$$

This case leads to the triple frequency ratio sets

$$\begin{aligned} \nu_\theta : (\nu_K - \nu_r) : \nu_r &= 4 : 3 : 2, \\ \nu_K : \nu_\theta : (\nu_K - \nu_r) &= 5 : 4 : 3, \end{aligned} \quad (51)$$

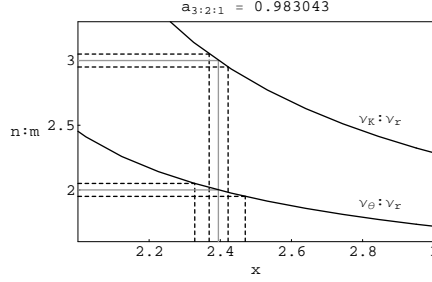


Fig. 2. Calculation of the width of the resonant radius where the strong resonant phenomena are possible. Black lines represent the ratio $n:m = v_K:v_r, v_\theta:v_r$, dashed lines illustrate the relative frequency detuning $\delta = 5\%$ (i.e., $n:m = 2 \pm 0.05$ and 3 ± 0.05). Gray solid lines represent the extraordinary case where the frequency ratio $v_K:v_\theta:v_r = 3:2:1$ arises at the same radius $x_{3:2:1} = 2.395$. We can simply calculate that at $x = 2.369$ the frequency ratio is $v_K:v_\theta:v_r = 3.05:2.019:1$ and at $x = 2.422$, there is $v_K:v_\theta:v_r = 2.95:1.981:1$.

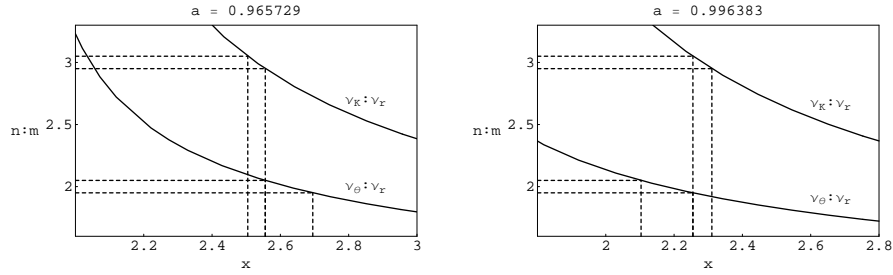


Fig. 3. Calculation of the width of the special value of the black hole spin parameter $a_{3:2:1}$ for which the multiple values frequencies should be observed. Black lines represent the ratio $n:m = v_K:v_r, v_\theta:v_r$, dashed lines illustrate the relative frequency detuning $\delta = 5\%$ (i.e., $n:m = 2 \pm 0.05$ and 3 ± 0.05). *Left panel:* The minimal value of the black hole spin parameter allowing the strong resonant phenomena where $v_K:v_\theta:v_r = 2.95:2.05:1$ at $x = 2.556$. *Right panel:* The maximal value of the black hole spin allowing the strong resonant phenomena where $v_K:v_\theta:v_r = 3.05:1.95:1$ at $x = 2.255$.

and again the related four frequency ratio sets

$$\begin{aligned} v_K : v_\theta : (v_K - v_r) : (v_r = v_\theta - v_r) &= 5 : 4 : 3 : 2, \\ v_K : v_\theta : (v_r = v_\theta - v_r) : (v_K - v_\theta) &= 5 : 4 : 2 : 1, \end{aligned} \quad (52)$$

and one five frequency ratio set

$$v_K : v_\theta : (v_K - v_r) : (v_r = v_\theta - v_r) : (v_K - v_\theta) = 5 : 4 : 3 : 2 : 1. \quad (53)$$

Considering the extraordinary resonant spin $a = a_{3:2:1} = 0.983$, we can conclude that the resonant phenomena are possible in some region around the resonant radius $x_{3:2:1} = 2.395$ due to the allowed frequency detuning with detuning parameter σ given by the frequency-response equations (see, e.g., Eq. 9, Eq. 12). The magnitude of the frequency scatter $\Delta = \epsilon\sigma$ is fully determined by a concrete resonance under consideration and by physical conditions in the oscillatory system, nevertheless, we can put a general restriction on the allowed frequency detuning as few percent of the oscillator eigenfrequency (Nayfeh and Mook 1979) and use it to find region of the disc around $x_{3:2:1}$, where the resonance could appear. We introduce the relations

$$v_1 = 2v_0 \pm \Delta, \quad v_2 = 3v_0 \pm \Delta, \quad (54)$$

$$\frac{t}{u} = \frac{v_1}{v_0} = 2 \pm \delta, \quad (55)$$

$$\frac{s}{u} = \frac{\nu_2}{\nu_0} = 3 \pm \delta, \quad (56)$$

where

$$\delta \equiv \frac{\Delta}{\nu_0}; \quad (57)$$

the resonance could appear where the regions given by conditions (55), (56) overlap (see Fig. 2). We find that the maximal frequency detuning of 0.05 implies possibility of the strong resonant phenomena in the interval of $x \in (2.369; 2.422)$ extended around the radius $x_{3:2:1} = 2.395$. Using this approach, we can give, for a given relative frequency detuning $\delta = \Delta/\nu_0$, the range of the black hole spin allowing the strong resonant phenomena. We simply find the spin where the overlap of the radial regions allowing resonance disappears (see Fig. 3). We choose characteristic values of $\delta = 0.01, 0.02$ and 0.05 and using a simple numerical code we determine the scatter around the special resonant spin values $a_{3:2:1}$, $a_{4:3:1}$ and $a_{5:3:1}$. The results are summarized in Table 1. For the other special resonant spins and any appropriately chosen value of δ , the spin scatter can be given by the numerical code. (We give the spin with precision of 0.001 as the 1% frequency scatter implies the spin scatter ≈ 0.001 .)

Table 1

Estimates of the spin value range where the strong resonant phenomena are possible for characteristic values of relative frequency detuning δ .

δ [%]	$s:t:u = 3:2:1$	$s:t:u = 4:3:1$	$s:t:u = 5:3:1$
1	$a_{3:2:1} = 0.983^{+0.003}_{-0.003}$	$a_{4:3:1} = 0.866^{+0.005}_{-0.005}$	$a_{5:3:1} = 0.962^{+0.001}_{-0.001}$
2	$a_{3:2:1} = 0.983^{+0.006}_{-0.006}$	$a_{4:3:1} = 0.866^{+0.010}_{-0.010}$	$a_{5:3:1} = 0.962^{+0.003}_{-0.003}$
5	$a_{3:2:1} = 0.983^{+0.013}_{-0.017}$	$a_{4:3:1} = 0.866^{+0.023}_{-0.027}$	$a_{5:3:1} = 0.962^{+0.006}_{-0.007}$

4 A Possible Application to Sgr A*

The resonant phenomena could occur frequently, when different versions of the resonance model can be realized at a shared radius (or its close vicinity) fixed by the frequency ratio $\nu_K : \nu_\theta : \nu_r = s : t : u$.

As explored above, there is an extraordinary resonant black hole spin $a_{3:2:1} = 0.983$, when at the radius $x_{3:2:1} = 2.395$ the frequency ratio is $\nu_K : \nu_\theta : \nu_r = 3 : 2 : 1$. Clearly, in vicinity of black holes with $a = 0.983$, the resonant phenomena should be strong, as the order of the resonances is of the lowest possible values, and, moreover, all the resonances, including those with beat frequencies, could cooperate efficiently even for frequencies slightly scattered from the exact resonant eigenfrequencies. The scatter of resonant frequencies strongly depends on the order of resonances and can be wide for resonances of very low order (Landau and Lifshitz 1976).

When the simple combinational frequencies could be considered, the frequency ratios, *e.g.*, $\nu_r : (\nu_K - \nu_\theta) = 1 : 1$, then allow strongest possible resonances.

That is the reason why we could expect well observable QPOs in the field of black holes with spin close to the values allowing the cooperating resonant phenomena sharing a fixed radius. It is important to look for some candidate systems exploring the simple triple frequency ratio sets. In such situations we can determine the black hole spin with high precision, given by the frequency measurement error (see Fig. 4). This

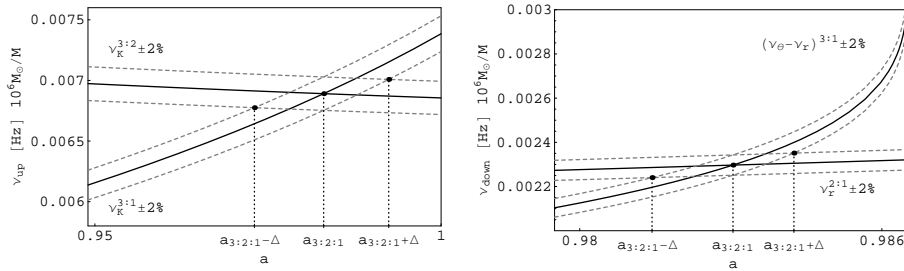


Fig. 4. Error in determining the extraordinary resonant spin $a_{3:2:1} = 0.983$: the scatter of the black hole spin related to the 2% error in frequency measurements. It has to be confronted with the spin scatter allowing occurrence of resonances (see Eqs (55)–(57)). *Left panel*: The case of top identity $v_{\text{upp}} = v_K$: the interval of allowed values of the spin is then $a \in \langle 0.973; 0.992 \rangle$. *Right panel*: The case of bottom identity $v_{\text{down}} = v_r = (v_\theta - v_r)$: the interval of allowed values of the spin is $a \in \langle 0.981; 0.984 \rangle$.

could help very much in determining the other physical characteristics of the systems. Notice that errors of frequency measurements imply some errors in the spin determination, as illustrated in Fig. 4. It depends also on the concrete resonances occurring at a given radius. Clearly, if the relevant frequency curves cross in a large (small) relative angle, the spin is determined with high (low) precision.

The Galaxy centre source Sgr A* can serve as a proper candidate system, since three QPOs were reported (but not fully accepted by the astrophysical community) for the system (Aschenbach 2004, Török 2005) with frequency ratio corresponding to the extraordinary resonant spin

$$(1/692) : (1/1130) : (1/2178) \approx 3 : 2 : 1 \quad (58)$$

and with the upper frequency being observed with a rather high error

$$v_{\text{upp}} = (1.445 \pm 0.16) \text{ mHz}. \quad (59)$$

Considering the standard epicyclic resonance model for the two upper frequencies $v_\theta : v_r = 3 : 2$, and the bottom frequency $v_{\text{down}} = v_\theta - v_r$, we obtain $v_\theta : v_r : (v_\theta - v_r) = 3 : 2 : 1$. Then identifying $v_{\text{upp}} = v_\theta$, we can show that for $a \leq 1$, the black hole mass $M \leq 2.3 \times 10^6 M_\odot$, which is in clear disagreement with the allowed range of the Sgr A* mass coming from the most recent analysis of the orbits of stars moving within 0.01 pc of Sgr A* (Ghez *et al.* 2008)

$$3.5 \times 10^6 M_\odot < M < 4.7 \times 10^6 M_\odot \quad (60)$$

and with the estimate $M \approx 4 \times 10^6 M_\odot$ obtained by Reid (2008).

If the black hole is assumed to be at rest with respect to the Galaxy (*i.e.*, has no massive companion to induce motion), the fit could be further constrained to the interval (Ghez *et al.* 2008)

$$4.1 \times 10^6 M_\odot < M < 4.9 \times 10^6 M_\odot \quad (61)$$

that is the most likely mass estimate at present, considering the discovery that Sgr A* is nearly stationary at the Galactic center (Reid 2008).

However, assuming a black hole with the spin comparable to the extraordinary resonant value of $a \approx 0.983$, with the frequency ratio $v_K : v_\theta : v_r = 3 : 2 : 1$ at the sharing radius $x_{3:2:1} = 2.395$, and identifying $v_{\text{upp}} = v_K$, we obtain the black hole mass of Sgr A* in the interval

$$4.29 \times 10^6 M_\odot < M < 5.36 \times 10^6 M_\odot \quad (62)$$

which meets the allowed black hole mass interval (Eq. 60) at its high mass end and is in good agreement with the estimate (Eq. 61).

5 Conclusions

We present the possibilities for strong resonant phenomena arising in a shared radius in the field of black holes with spin appropriately tuned. We assume that the strong resonances could occur and influence each other causally, if $v_K : v_\theta : v_r$ is in ratio of small integers $s : t : u$ with $s \leq 5$, when the order of the resonances could be low enough in order to enable strong resonance with relatively wide resonance frequency width (Landau and Lifshitz 1976). The strong resonances with lowest values of the frequency ratio occur for the extraordinary resonant spin $a_{3:2:1} = 0.983$.

Due to the allowed frequency detuning in the resonant phenomena (with maximal relative frequency detuning limited to few percent), the strong resonant phenomena could appear in black holes with spin values concentrated around the special resonant spin values. For a given relative frequency detuning δ , the spin scatter differs for different special resonant spin values, being dependent on the position of the resonant radius (and the gradient of the frequency profiles at this radius), see Table 1. In the most interesting case of the extraordinary resonant spin $a_{3:2:1} = 0.983$, the spin scatter $\approx \pm 0.003$ for detuning $\delta = 0.01$ gives probably a realistic interval of black hole spins where the strong resonant phenomena could be observable. Of course, the value of δ (and the spin value scatter) depend on physical conditions and resonant phenomena in concrete sources and have to be discussed carefully for each individual source.

There is an indication that the QPOs data observed in the Galactic centre source Sgr A* imply the black hole spin close to the extraordinary resonant value of $a \approx 0.983$, when $v_K : v_\theta : v_r = 3 : 2 : 1$ holds at the radius $x_{3:2:1} = 2.395$ and the black hole mass can be estimated to be in the interval of $(4.8 \pm 0.5) \times 10^6 M_\odot$. Therefore, it is interesting to check concordance of the QPOs observation induced data with a variety of the other observations of the Sgr A*. It should be stressed that more precise measurement of the QPOs frequencies will enable more precise determination of the black hole mass. Moreover, we expect NGC 5408 X-1 to be another candidate for a black hole admitting strong resonant phenomena because of the observed frequency set (Strohmayer *et al.* 2007).

It is important from the principal reason to note that if a black hole with appropriately tuned spin a admits strong resonant phenomena with different resonances sharing the same radius of the accretion disc, we could expect observable QPOs at resonant frequencies even in situations when the internal disc conditions are not convenient for starting up resonant phenomena, as observed, *e.g.*, in Sgr A* source (Genzel *et al.* 2003).

On the other hand, the four (or five) QPOs frequencies observed in the microquasar GRS 1915+105 can not be explained by the strong resonances model because of specific distribution of the observed frequency ratios (two different pairs of frequencies with 3:2 ratio). In this case, the extended resonant model including the so called humpy oscillations induced by the “humpy” velocity profile of the accretion disc, as measured in the privileged family of locally non-rotating frames, can explain all the observed frequencies, if the black hole parameters are fixed at the spin $a \approx 0.9998$ and the mass $M \approx 14.4 M_\odot$ (Stuchlík *et al.* 2007c), in agreement with the spectral fit of the spin $a \approx 1$ (McClintock *et al.* 2006).

The conditions for strong resonant phenomena could be realized only for black holes with high values of dimensionless spin ($a \geq 0.75$). Therefore, the idea of strong resonant phenomena probably could not be extended to the neutron star systems, where we expect spin $a < 0.5$, at least if the Hartle–Thorne metric parameters corresponding to the spin a and the quadrupole momentum q are close to the quasi-Kerr values with $q \sim j^2$. Note that for the neutron star systems probably another version of the multiresonant idea could be realized in close agreement with observational data, where the so called

total precession resonance model with resonant frequencies ν_K and $\nu_\theta - \nu_r$ (related to the relativistic precession model, Stella and Vietri 1998) is involved and realized at different radii, as shown in Stuchlík *et al.* (2007e).

In the field of Kerr black holes with $a \approx 0.866$, the four frequency set ratio 4 : 3 : 2 : 1 could be observed, if the resonant conditions allow combinational frequencies to be observable. For the spin $a \approx 0.962$ or $a \approx 0.882$, even the five frequency set 5 : 4 : 3 : 2 : 1 could be observable in some special circumstances. Clearly, it is not necessary that all the resonances are realized simultaneously and that the full five frequency set is observed at the same time.

Undoubtedly, it is worth to search for such special cases of frequency sets in observational data as these could lead to precise determination of the black hole spin enabling a deeper understanding of a variety of astrophysical phenomena involved in determining behaviour of accretion discs in strong gravitational fields. We should note that when simple combinational frequencies not exceeding the Keplerian frequency are allowed, the lowest triple frequency ratio set 3 : 2 : 1 can be realized for black holes with the extraordinary resonant spin $a \approx 0.983$, but also for the black holes with three other values of the spin $a = 0.866, 0.882, 0.962$, if the uppermost frequencies are not observed for some reasons.

Finally, it has to be stressed that observation of some characteristic frequency sets with low integer ratio could be a signature of a specific value of the black hole spin a enabling strong resonances at a fixed radius. However, generally, there exist few values of the spin a and the corresponding shared resonance radius allowed for a given frequency ratio set (Stuchlík *et al.* 2007b). Therefore, detailed analysis of the resonance phenomena, including the resonance strength and resonant frequency width, has to be considered in a concrete candidate system, and it has to be further confronted with the spin estimates coming from spectral analysis of the black hole system as given, *e.g.*, in McClintock *et al.* (2006) and Middleton *et al.* (2006) for GRS 1915+105, and Shafee *et al.* (2006) for GRO J1655-40, the line profiles (Fabian and Miniutti 2005, Dovčiak *et al.* 2004, Zakharov 2003, Zakharov and Repin 2006), and the orbital periastron precession of some stars moving in the region of Sgr A* (Kraniotis 2005, 2007), in order to establish the black hole spin. Very promising from this point of view seem to be studies of the energy dependencies of high-frequency QPO determining the QPO spectra at the QPO radii (Życki *et al.* 2007) as they could be credited by the strong resonance conditions giving precisely the black hole spin and the radius where the observed QPOs are generated.

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