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RELATIVISTIC DYNAMICS WITH COSMOLOGICAL CONSTANT: SPINNING TEST PARTICLES AND PERFECT FLUID TORI

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Results of our recent studies concerning possible effects of $\Lambda > 0$ for equilibrium positions of spinning test particles and stationary configurations of perfect-fluid tori are presented.

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1. Introduction

In this paper, we continue in presentation of our results concerning effects of $\Lambda > 0$ in astrophysically motivated problems started in Paper I in this issue[1], discussing examples of non-geodesic motion. Here, we summarize basic results obtained in the studies of equilibrium positions of spinning test particles[2,3] and stationary toroidal configurations of a barotropic perfect fluid[4,5] in the Schwarzschild-de Sitter (SdS) and Kerr-de Sitter (KdS) spacetimes. Compared to the spinless particles, the motion equation of spinning test particles is more complex, because of the interaction of the spin with the curvature of the spacetime given by the Riemann tensor. Moreover, the spin dynamics has also to be considered. In the perfect-fluid tori, pressure gradients are relevant. The KdS spacetime is characterized by three parameters: M (mass), a (rotational parameter) and $y = \Lambda M^2/3$ (cosmological parameter). As in the previous paper, we use Boyer-Lindquist coordinates (t, r, θ, φ) and dimensionless formulation given by $c = G = M = 1$.

2. Spinning test particles equilibrium conditions

Motion of a spinning test particle of mass m, 4-velocity u^λ, and spin S^λ in a gravitational field is given by the equation

$$m\frac{Du^\alpha}{d\tau} = -\epsilon^{\alpha\mu\nu\beta}\frac{D^2u_\beta}{d\tau^2}S_\mu u_\nu + \frac{1}{2}\epsilon^{\lambda\mu\rho\sigma}R^\alpha_{\nu\lambda\mu}u^\nu u_\sigma S_\rho, \tag{1}$$

where the covariant Pirani spin vector $S_\sigma = \frac{1}{2}\epsilon_{\rho\mu\nu\sigma}u^\rho S^{\mu\nu}$; $S^{\mu\nu}$ is the particle spin tensor, $\epsilon_{\rho\mu\nu\sigma}$ is the Levi-Civita tensor, and $D/d\tau$ denotes the covariant derivate along the vector field u^α, i.e. $Du^\alpha/d\tau = u^\beta(\partial_\beta u^\alpha + \Gamma^\alpha_{\beta\gamma}u^\gamma)$. The spin of a particle is orthogonal to its 4-velocity, i.e., $S^\alpha u_\alpha = 0$, and dynamics of the spin vector is given by the Fermi-Walker transport equation

$$\frac{DS_\alpha}{d\tau} = u_\alpha \frac{Du^\beta}{d\tau} S_\beta. \tag{2}$$

The 4-velocity of a *test particle at rest* can be written as

$$u^\alpha = \frac{1}{\sqrt{-g_{tt}}}\delta^\alpha_t, \quad \frac{du^\alpha}{d\tau} = u^\beta \partial_\beta u^\alpha = 0, \tag{3}$$

which represents equilibrium condition for static particles. Clearly, the equilibrium of such particles is possible only outside ergospheres of the spacetime, because in the ergosphere, there is $g_{tt} > 0$, and particles must corotate with the spacetime.

The orthogonality of the spin and the 4-velocity implies that $S_t = 0$, i.e., there are only space components of the spin vector. Investigation of Eqs. (1) and (2) under these conditions implies following results.

Due to the combined effect of the rotation of the spacetime and the cosmic repulsion, the equilibrium of spinning particles with 4-velocity (3) is spin-dependent in the KdS spacetime, in contrast to the spherically symmetric static SdS spacetime where the equilibrium is possible only at the *static radius* $r = r_{\mathrm{s}} \equiv y^{-1/3}$ independently of the spin, i.e., even for spinless particles. Of course, in the Schwarzschild spacetime, the equilibrium is not possible at all. In the equatorial plane of the KdS spacetime, the equilibrium is possible at the static radius for both the spinless particles as well as for spinning particles with arbitrarily large φ-oriented spin, or at any radius for spinning particles with the spin orthogonal to the equatorial plane satisfying the condition

$$S_\theta = m\frac{r^2(1-yr^3)[2r - r^2 + yr^2(r^2+a^2)]}{a[y^2r^6 + y(r^3 + 3a^2r) - 3r + 7](1+ya^2)}. \tag{4}$$

In the Kerr spacetime, there is no equilibrium for spinless particles in the equatorial plane. On the other hand, spinning particles can stay at rest at any radius outside the ergosphere, if their spin vector is orthogonal to the equatorial plane and satisfies the condition (4) for $y = 0$. Situation on the symmetry axis of the Kerr spacetime is decribed in the paper of Stuchlík and Kovář.[3]

3. Toroidal configurations of barotropic perfect fluid

Dynamics of a perfect fluid orbiting in the stationary and axisymmetric spacetime is described by the relativistic Euler equation. For the fluid with the 4-velocity field $u^\mu = (u^t(r,\theta), 0, 0, u^\varphi(r,\theta))$, it has the form:[6]

$$\frac{\partial_k p}{\epsilon + p} = -\partial_k(\ln u_t) + \frac{\Omega\partial_k \ell}{1 - \Omega\ell}, \quad k = \{r,\ \theta\} \tag{5}$$

where

$$(u_t)^2 = \frac{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}{g_{tt}\ell^2 + 2g_{t\varphi}\ell + g_{\varphi\varphi}}; \tag{6}$$

$\Omega(r,\,\theta) = u^\varphi/u^t$ and $\ell(r,\,\theta) = -u_\varphi/u_t$ describe distribution of the angular velocity and the specific angular momentum in the fluid. Note that within the inertial forces formalism, the relativistic Euler equation for the rotating perfect fluid can be written in the 'Newtonian' form

$$\frac{\partial_k p}{\epsilon + p} = -F_k{}^{\perp}, \quad k = \{r,\ \theta\} \tag{7}$$

where $F_k^{\perp} = G_k + Z_k + C_k + E_k$ is the sum of inertial forces.[1,7] Integration of Eq. (5) or (7) for the barotropic fluid enables to define the potential $W = W(r,\,\theta)$, which equipotential surfaces coincide with the surfaces of constant pressure in the fluid. Orbits where $\partial_k W(r,\,\theta) = 0$ correspond to free-particle orbits (geodesics), because the pressure-gradient forces are zero there. In the simpliest case of uniformly distributed specific angular momentum, $\ell(r,\,\theta) = \text{const}$, the potential W is given by the relation $W = \ln u_t$, which in the KdS spacetime takes the form[5]

$$W(r,\,\theta) = \ln\left\{\frac{\rho^2}{I^2}\frac{\Delta_r\Delta_\theta\sin^2\theta}{\Delta_\theta(r^2+a^2-a\ell)^2\sin^2\theta - \Delta_r(\ell - a\sin^2\theta)^2}\right\}^{1/2}. \tag{8}$$

Equipotential surfaces can be closed (toroidal) or open (cylindrical). Moreover, there is a special class of *critical surfaces* self-crossing in the cusp(s), which can be either marginally closed or open. Closed equipotential surfaces determine stationary equilibrium configurations (tori).[a] In the center of any torus, the pressure attains an extreme value (maximum) and matter must follow a stable geodesic there. Thus, fluid tori can exist only in the spacetimes with stable circular geodesics.

In the KdS (SdS) spacetime, three qualitatively different types of tori can exist, depending on the character of the critical marginally closed equipotential surface (Fig. 1). This surface enables an outflow from the torus through the cusp(s) in the equatorial plane, when the surface of the disc overcomes the critical surface. The outflow is thus driven by a violation of hydrostatic equilibrium in the torus, rather than by a viscosity of the fluid. The first type of configurations corresponds to the well known *accretion discs*. The cusp of the critical marginally closed equipotential surface is located at the inner edge of torus between the inner marginally stable and the inner marginally bound circular geodesics of the spacetime, $r_{\rm mb(i)} < r_{\rm in} < r_{\rm ms(i)}$. Moreover, there is another critical surface, self-crossing in the outer cusp, which is open. The second type of configurations is the so-called *excretion disc*. The cusp of the critical marginally closed equipotential surface is located at the outer edge of the torus between the outer marginally stable and the outer marginally bound circular geodesics, $r_{\rm ms(o)} > r_{\rm out} > r_{\rm mb(o)}$. The second critical surface, self-crossing

[a]Topological properties of equipotential surfaces, in general, seem to be rather independent of the distribution of the specific angular momentum $\ell(r,\,\theta)$.

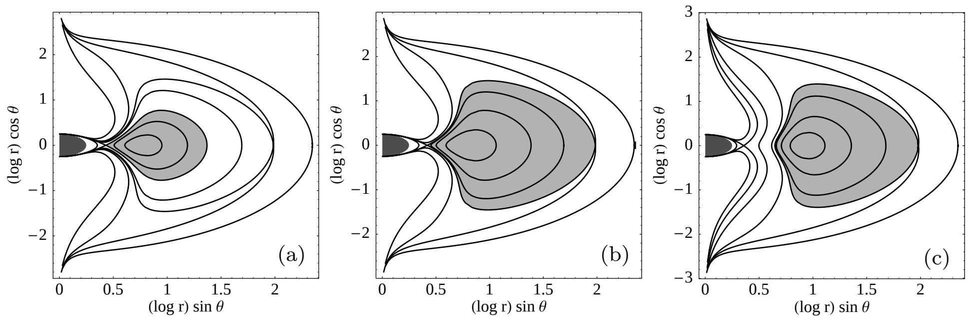

Fig. 1. Equilibrium toroidal configurations of perfect fluid in SdS and KdS black-hole spacetimes: (a) accretion discs, (b) marginally bound accretion discs, (c) excretion discs.

in the inner cusp, is open. Due to the existence of other open equipotential surfaces between the inner and outer critical surfaces, the outflow, i.e. accretion, onto the central object is not possible. In the special case of the so-called *marginally bound accretion disc*, both the cusps belong to the same critical equipotential surface, and their locations coincide with the inner and outer marginally bound geodesics, $r_{\rm in} = r_{\rm mb(i)}$, $r_{\rm out} = r_{\rm mb(o)}$. Therefore the marginally bound accretion disc is the most extended torus in given KdS spacetime. Table 1 shows radii of the outer marginally stable orbit and the static radius in kiloparsecs for the current value of the cosmological constant $\Lambda_0 = 1.3 \times 10^{-52}\,\mathrm{m}^{-2}$ (in units $c = G = 1$) and extreme KdS black holes. Note that for $\Lambda = \Lambda_0$, $r_{\rm mb(o)} \approx r_{\rm s}$.

Table 1.

y	10^{-44}	10^{-42}	10^{-40}	10^{-34}	10^{-32}	10^{-30}	10^{-28}
$M/M_\odot$	10	10^2	10^3	10^6	10^7	10^8	10^9
$r_{\rm ms(o)}$/kpc	0.15	0.31	0.67	6.7	15	31	67
$r_{\rm s}$/kpc	0.23	0.50	1.1	11	23	50	110

Acknowledgments

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